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Markov-Type Inequalities and the Degree of Convex Spline Interpolation

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1. INTRODUCTION

It has been shown by Passow and Roulier [8] and by McAllister and Roulier [6] that it is impossible to interpolate convex data by a smooth, convex, piecewise polynomial with fixed knots and bounded degree, independent of the data and knots. Indeed, a lower bound for this degree is given in [6] and [8] which shows that a suitable choice of the y-coordinates of the data points can make the degree as large as desired.

In this paper, we show that the lower bound mentioned above is best possible. This is accomplished by observing that the classical Markov inequality cannot be improved by restricting consideration to convex increasing polynomials.

2. NOTATION AND BACKGROUND

For each nonnegative integer n, let Π_n denote the set of *algebraic polynomials of degree n or less*. For such a given n, a mesh

$$\Delta = \{t_0, t_1, \ldots, t_M\}$$

with

$$t_0 < t_1 < \cdots < t_M,$$

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and a nonnegative integer $m \leq n$, consider the set of nth-degree splines of deficiency $n - m$ with knots Δ:

$$S_n^m(\Delta) = \{f \in C^m[t_0, t_M]: f \in \Pi_n \text{ on } [t_{i-1}, t_i], i = 1,..., M\}.$$

Let data $\{(x_i, y_i)\}_{i=0}^N$ with $x_0 < \cdots < x_N$ be given. The data is *increasing* and *convex* if $0 < S_1 < S_2 < \cdots < S_N$, where

$$S_i = \frac{y_i - y_{i-1}}{x_i - x_{i-1}}, \qquad i = 1,..., N.$$

If $f \in S_n^1(\Delta)$ is convex and increasing on $[x_0, x_N]$ and satisfies $f(x_i) = y_i$, $i = 0, 1,..., N$, then a lower bound for n can be found in terms of the slopes $\{S_i\}_{i=1}^N$.

Passow and Roulier [8] show such a lower bound for $\Delta = \{-1, 1, 3, 5\}$ and McAllister and Roulier [6] generalize this result to arbitrary fixed knots Δ which need not consist entirely of interpolation abscissas. We present the latter result here.

THEOREM 2.1. *Let mesh Δ and convex, increasing data (x_i, y_i), $i = 0, 1, 2, 3$, be given. If $f \in S_n^1(\Delta)$ satisfies*

$$f(x_i) = y_i, \qquad i = 0, 1, 2, 3, \tag{2.1}$$

and

$$f \text{ is convex and increasing on } [x_0, x_3], \tag{2.2}$$

then

$$n^2 \geq \frac{S_2}{\dfrac{(x_2 - x_1)(S_3 - S_2)}{\mu - x_1} - \dfrac{(x_1 - x_0)S_1}{x_1 - l}}, \tag{2.3}$$

where $l = \max\{x \in \Delta: x < x_1\}$ and $\mu = \min\{x \in \Delta: x > x_1\}$.

We note that if $\Delta = \{x_0, x_1, x_2, x_3\}$ then $l = x_0$ and $\mu = x_2$ and (2.3) becomes

$$n^2 \geq \frac{S_2}{S_3 - S_2 - S_1}. \tag{2.4}$$

It is then shown in [6] and [8] that a suitable choice of y_0, y_1, y_2, y_3 will force n to be as large as desired in (2.3) (and (2.4)).

3. MARKOV INEQUALITIES

The classical Markov inequality states that if $p \in \Pi_n$ and if $\|\cdot\|$ is the sup norm on $[a, b]$, then

$$\|p'\| \leq \frac{2n^2}{b - a} \|p\|. \tag{3.1}$$

Moreover, this inequality cannot be improved. For example, the Chebyshev polynomials actually give equality in (3.1). Erdös [3] has shown that (3.1) can be improved by restricting the class of polynomials. That is, n^2 in (3.1) can be replaced by $en/2$ for polynomials with only real zeros not in $[a, b]$.

The purpose of this section is to show that the exponent 2 in (3.1) cannot be reduced by restricting consideration to the class of convex, increasing polynomials. This result will be used in the following section to produce the theorem alluded to in Section 1.

We proceed to the first result. Let p_n be the nth degree Legendre polynomial and define

$$s_n(x) = \int_0^x \int_0^\mu (p_n''(t))^2 \, dt \, d\mu. \tag{3.2}$$

We note that

$$s_n \in \Pi_{2n-2},$$
$$s_n(0) = 0,$$
$$s_n \text{ is convex and increasing on } [0, 1].$$

Moreover, if we let $\|\cdot\|$ be the sup norm on [0, 1], then it can be shown that

$$\| s_n' \| \geqslant \frac{n^2}{6} \| s_n \|. \tag{3.3}$$

One proves (3.3) by observing that

$$\| s_n \| = s_n(1)$$

and

$$\| s_n' \| = s_n'(1).$$

Then show that

$$s_n'(1) = p_n''(1)\, p_n'(1) - p_n'''(1)$$
$$p_n''(1) \leqslant s_n(1) \leqslant 2p_n''(1).$$

The following are either well known or follow easily by successively differentiating the well-known differential equation (see [9])

$$(1 - x^2)\, p_n''(x) - 2xp_n'(x) + n(n + 1)\, p_n(x) = 0.$$
$$p_n(1) = 1$$
$$p_n'(1) = \frac{n(n+1)}{2}$$
$$p_n''(1) = \frac{n(n+1) - 2}{4} p_n'(1) = \left(\frac{n(n+1) - 2}{4}\right)\left(\frac{n(n+1)}{2}\right)$$
$$p_n'''(1) = \frac{n(n+1) - 6}{6} p_n''(1).$$

It then follows from these that

$$\frac{s_n'(1)}{s_n(1)} \geq \frac{n(n-1)}{6} \geq \frac{1}{2}.$$

Thus, (3.3) follows.
But $s_n \in \Pi_{2n-2}$. Thus, it follows from (3.3) that given any interval $[a, b]$, there is a sequence of polynomials $\{q_k\}_{k=0}^{\infty}$, $q_k \in \Pi_k$, such that q_k are nonnegative, convex, and increasing on $[a, b]$, $q_k(a) = 0$, and

$$q_k' \geq \frac{k^2}{24(b-a)} q_k .$$

That is if $k = 2n - 1$ or $2n - 2$ we define $q_k(x) = s_n((x - a)/(b - a))$. This leads us to state

THEOREM 3.1. *Markov's inequality* (3.1) *cannot be improved by replacing n^2 by some lower power of n for convex increasing polynomials.*

4. CONVEX, INCREASING SPLINES

We now use the results of the previous section to show that in the general case the estimate (2.4) cannot be improved by replacing n^2 by some lower power of n.

THEOREM 4.1. *Let $\Delta = \{x_0, x_1, x_2, x_3\}$ be given. Then for each positive integer n there exists $f_n \in S_n^1(\Delta)$ convex and increasing on $[x_0, x_3]$ so that $\{(x_i, f_n(x_i))\}_{i=0}^{3}$ is convex, increases and*

$$\frac{S_{2,n}}{S_{3,n} - S_{2,n} - S_{1,n}} \leq n^2 \leq \frac{48 S_{2,n}}{S_{3,n} - S_{2,n} + S_{1,n}}, \tag{4.1}$$

where

$$S_{i,n} = \frac{f_n(x_i) - f_n(x_{i-1})}{x_i - x_{i-1}} \qquad i = 1, 2, 3.$$

Proof. Let n be given. If $n = 1$, the problem is trivial. So, assume that $n > 1$. We construct $f_n \in S_n^1(\Delta)$ satisfying the second inequality of (4.1). The first inequality follows from (2.4). To perform this construction we make use of (3.4). By (3.4), we choose $q_n \in \Pi_n$ convex and increasing on $[x_0, x_1]$ such that

$$q_n(x_0) = 0$$

and

$$\max_{x_0 \leq x \leq x_1} |q_n'(x)| > \frac{n^2}{24(x_1 - x_0)} \max_{x_0 \leq x \leq x_1} |q_n(x)|. \tag{4.2}$$

Now define

$$\begin{aligned} f_n(x) &= q_n(x), & x &\leq x_1 \\ &= q_n(x_1) + q_n'(x_1)(x - x_1) + \frac{S_{1,n}}{x_3 - x_2}(x - x_2)_+^2, & x &> x_1 . \end{aligned}$$

Then $f_n \in S_n{}^1(\Delta)$ is increasing and convex, and

$$S_{2,n} = q_n'(x_1) > S_{1,n} > 0,$$

and so

$$S_{3,n} = q_n'(x_1) + S_{1,n} = S_{2,n} + S_{1,n} > S_{2,n} .$$

This shows that $\{(x_i, f_n(x_i))\}_{i=1}^3$ is increasing convex and that

$$\frac{S_{2,n}}{S_{3,n} - S_{2,n} - S_{1,n}} = \frac{S_{2,n}}{2S_{1,n}} .$$

On the other hand, by (4.2),

$$\begin{aligned} S_{2,n} = q_n'(x_1) &= \max_{x_0 \leq x \leq x_1} |q_n'(x)| > \frac{n^2}{24(x_1 - x_0)} \max_{x_0 \leq x \leq x_1} |q_n(x)| \\ &= \frac{n^2}{24(x_1 - x_0)} (q_n(x_1) - q_n(x_0)) = \frac{n^2}{24} S_{1,n} \end{aligned}$$

and so

$$n^2 \leq \frac{48 S_{2,n}}{S_{3,n} - S_{2,n} + S_{1,n}} .$$

This proves the theorem.

The same approach can be used to show that the more general estimate (2.3) cannot be improved either.

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